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Aspects of Higher Curvature Terms and U-Duality

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Abstract: We discuss various aspects of dimensional reduction of gravity with the Einstein–Hilbert action supplemented by a lowest order deformation formed as the Riemann tensor raised to powers two, three or four. In the case of R^2 we give an explicit expression, and discuss the possibility of extended coset symmetries, especially $SL(n+1, \mathbb{Z})$ for reduction on an n -torus to three dimensions. Then we start an investigation of the dimensional reduction of R^3 and R^4 by calculating some terms relevant for the coset formulation, aiming in particular towards $E_{8(8)}/(Spin(16)/\mathbb{Z}_2)$ in three dimensions and an investigation of the derivative structure. We emphasise some issues concerning the need for the introduction of non-scalar automorphic forms in order to realise certain expected enhanced symmetries.

1. INTRODUCTION AND SUMMARY

M-theory, when compactified on an n -torus, is conjectured to have a global U-duality symmetry $E_{n(n)}$ in the low-energy limit described by maximal supergravity in $d = 11 - n$ dimensions. It is known from string theory that this continuous symmetry is broken in the quantum theory to a discrete version $E_{n(n)}(\mathbb{Z})$. The massless scalars in the compactified theory belong to the coset $E_{n(n)}(\mathbb{Z}) \backslash E_{n(n)} / K(E_{n(n)})$, where $K(E_{n(n)})$ is the (locally implemented) maximal compact subgroup of the split form $E_{n(n)}$. When $d \leq 3$, no local massless bosonic degrees of freedom remain except scalar ones. It has been proposed that it may even be possible to define M-theory itself as a theory on the coset obtained when going to $d = 1$ (E_{10}) or $d = 0$ (E_{11}), although it is unclear whether or not such a formulation incorporates degrees of freedom beyond supergravity.

Some aspects of these discrete symmetries are well investigated. This concerns primarily calculations in cases with low dimension of the torus. For $n < 3$, non-perturbative string theory results are obtained from loop calculations in $D = 11$ supergravity. For $n \geq 3$ one expects that there will be contributions also from membrane instantons and for $n \geq 6$ from five-brane instantons. This makes results for higher-dimensional tori harder to obtain. On the other hand, one may turn the argument around and ask what kind of restrictions U-duality puts on the possible quantum corrections of the theory. It is convenient to work in the massless sector, obtained by dimensional reduction, and let quantum effects manifest themselves in an effective action, which will then contain higher orders of curvatures (and other fields), *i.e.*, higher-derivative terms.

Some partial results have been obtained by investigating the general structures of higher-derivative terms to determine if they can be made to fit into something U-duality invariant. For example it has been shown that the Riemann tensor in $D = 11$ comes only in powers $3k + 1$, where k is integer. The purpose of the present paper is to initiate a more detailed analysis aiming at actually checking the invariance. The scope of the paper is modest; we restrict our attention to the D -dimensional gravitational sector alone. Then we set out to form higher-derivative corrections to the Einstein–Hilbert action in the form of second, third and fourth powers of the Riemann tensor. The full U-duality group is not accessible with gravity only, but on compactification to $d = 3$ there still has to be an enhancement from $SL(8)$ to $SL(9)$, which is the subgroup of $E_{8(8)}$ of which the gravitational scalars form a coset (more generally, on reduction from $n + 3$ to 3 dimensions, we expect an enhancement from $SL(n)$ to $SL(n+1)$). Some aspects about the general structures of the higher curvature terms at hand are investigated, before we turn to examining chosen subsets of terms and thereby extracting concrete information concerning the possibility of implementing $SL(n + 1)$. We draw some definite conclusions about the necessity of introducing transforming automorphic forms, and show that they can always be chosen to reproduce the results in the dimensionally

reduced theory. The interpretation of the dimensionally reduced actions is not as U-duality invariant object *per se*, but as properly taken large volume limits of U-duality invariant actions involving transforming automorphic forms. The investigation is very much a partial one, and we point out some further directions, such as a more complete expansion in fields, and a concrete examination of cosets and discrete groups based on exceptional groups.

We refer to refs. [1,2] for an overview of U-duality. Topics on E_{10} and E_{11} as fundamental symmetries are dealt with in refs. [3,4,5,6] and references therein. Recent developments concerning the connections between U-duality and higher curvature terms are found in refs. [7,8,9,10]. For different approaches to higher curvature terms in supergravity and string theory, see refs. [11-17]. The $3k + 1$ restriction on powers of the Riemann tensor in eleven-dimensional supergravity is discussed in ref. [18].

2. THE TORUS DIMENSIONAL REDUCTION PROCEDURE

Our Ansatz for dimensional reduction on an n -torus to three dimensions is given by

$$\hat{E}^a = e^{-\phi} e^a, \quad \hat{E}^i = (dy^\mu - A^\mu) e_\mu{}^i. \quad (2.1)$$

Here, e^ϕ is not an independent field, but the determinant of the internal vielbein $e_\mu{}^i$. The prefactor $e^{-\phi}$ is chosen so that a canonically normalized Einstein–Hilbert term results in three dimensions from the reduction of such a term in the higher-dimensional theory. Our conventions are such that $D = d + n$ with D the space-time dimension before the dimensional reduction and d the one after, with n the dimension of the internal torus on which we are performing the dimensional reduction. Flat indices are denoted a, b, \dots in space-time and i, j, \dots on the internal manifold which is parametrised by coordinates y^μ . The one-forms A^μ in the above Ansatz are the n graviphoton potentials while $e_\mu{}^i$ is the internal vielbein and hence an element of $GL(n)$. One of our goals will be to see if this global symmetry (or, strictly speaking, $SL(n, \mathbb{Z})$, the mapping class group of the internal torus) is extended to larger groups when considering Lagrangians which consist of the Einstein–Hilbert term plus terms containing the Riemann tensor raised to powers 2, 3 and 4. This issue has previously been investigated by the authors of ref. [9] where the root and weight structure of the scalar prefactors arising in the reduction are studied. These prefactors are in ref. [9] extracted by applying some general arguments about the properties of higher derivative terms. In a continued work [10] they conclude that when weights instead of roots occur in the scalar exponent prefactors this should be compensated for by tensorial automorphic forms. The results obtained here by explicitly computing some of the relevant terms in the dimensional

reduction lend further support to such a construction. Automorphic forms of $SL(2, \mathbb{Z})$ with similar non-trivial properties have already been seen to arise in the type IIB superstring multiplying a term containing the product of 16 dilatinos [19].

From the above Ansatz one easily obtains, using the zero torsion condition, the dimensionally reduced form of the spin connection one-form and from it the Riemann tensor two-form. By reading off the components of these tensors using the basis indicated by the Ansatz above, *i.e.*, $\hat{e}^a = e^a$, $\hat{e}^i = (dy^\mu - A^\mu)_\mu{}^i$, we get an answer without explicit graviphoton potentials since this basis is manifestly translation invariant on the torus [20]. In order to examine the possibility of symmetry enhancement in reduction to $d = 3$, we need the following expressions for the components of the Riemann tensor

$$\begin{aligned}
\hat{R}_{ab}{}^{cd} &= e^{2\phi} [R_{ab}{}^{cd} + 4\delta_{[a}^{[c} D_{b]} \phi D^{d]} \phi + 4\delta_{[a}^{[c} D_{b]} \phi D^{d]} \phi - 2\delta_{ab}^{cd} D_e \phi D^e \phi] \\
&\quad - e^{4\phi} \left[\frac{1}{2} (F_{ab} F^{cd}) + \frac{1}{2} (F_{[a}{}^c F_{b]}{}^d) \right], \\
\hat{R}_{ab}{}^{cl} &= e^{3\phi} \left[\frac{1}{2} D^c F_{ab}{}^l + D^c \phi F_{ab}{}^l - D_{[a} \phi F_{b]}{}^{cl} + \delta_{[a}^c D^d \phi F_{b]d}{}^l \right. \\
&\quad \left. + \frac{1}{2} (F_{ab} P^c)^l + (F_{[a}{}^c P_{b]})^l \right], \\
\hat{R}_{ab}{}^{kl} &= -2e^{2\phi} (P_{[a} P_{b]})^{kl} - \frac{1}{2} e^{4\phi} F_a{}^{c[k} F_{bc}{}^{l]}, \\
\hat{R}_{aj}{}^{cl} &= +\frac{1}{4} e^{4\phi} F^{ce}{}_j F_a{}^l - e^{2\phi} [D_a P^c + D_a \phi P^c + D^c \phi P_a - \delta_a^c D_e \phi P^e + P_a P^c]_j{}^l, \\
\hat{R}_{aj}{}^{kl} &= -e^{3\phi} F_a{}^{e[k} P_{ej}{}^{l]}, \\
\hat{R}_{ij}{}^{kl} &= -2e^{2\phi} (P_e)_{[i}{}^k (P^e)_{j]}{}^l,
\end{aligned} \tag{2.2}$$

where $F_{ab}^i := F_{ab}^\mu e_\mu{}^i$, with $F_{mn}^\mu = 2\partial_{[m} A_{n]}^\mu$, are the graviphoton field strengths. We use the notation $(AB) = A^i B^i$ for the scalar product of $SO(n)$ vectors. The covariant derivative is $D_m = \partial_m + \omega_m + Q_m$. We have also defined P and Q as the symmetric and antisymmetric parts of the Maurer–Cartan one-form constructed from the internal vielbein $e_\mu{}^i$ (remember that they form the Maurer–Cartan form of $GL(n)$, so that $\text{tr} P = d\phi$). Q belongs to the $so(n)$ subalgebra and P spans the tangent directions of the corresponding coset $GL(n)/SO(n)$. As a direct consequence of their definition P and Q satisfy

$$DP := dP + PQ + QP = 0, \quad F_Q := dQ + Q^2 = -P^2. \tag{2.3}$$

We also have that the graviphotons satisfy the Bianchi identity $DF - F \wedge P = 0$.

Reduction of the D -dimensional Einstein–Hilbert term using these expressions leads directly to the following Lagrangian in $d = 3$:

$$\hat{E}\hat{R} = e [R - \text{tr}(P_a P^a) - \frac{1}{4} e^{2\phi} (F_{ab} F^{ab}) - D_a \phi D^a \phi], \tag{2.4}$$

where one should keep in mind that there is a hidden contribution to the kinetic term of the dilaton ϕ in the $GL(n)$ coset term. Note, however, that even after putting the two singlet terms together the kinetic term is not conventionally normalized in our conventions; see below for further details. The equations of motion that we will need in the following are (in $d = 3$)

$$\begin{aligned} R_{ab} &= \text{tr}(P_a P_b) + D_a \phi D_b \phi + \frac{1}{2} e^{2\phi} [(F_a{}^c F_{bc}) - \frac{1}{2} \eta_{ab} (F^{cd} F_{cd})] , \\ (D^a F_{ab})^i &= -(P^a F_{ab})^i - 2 D^a \phi F_{ab}^i , \\ (D^a P_a)^{ij} &= \frac{1}{4} e^{2\phi} F_{ab}^i F^{jab} . \end{aligned} \quad (2.5)$$

Note that the equation of motion for ϕ , $D^a D_a \phi = \frac{1}{4} e^{2\phi} (F^{ab} F_{ab})$, follows directly from the last equation above since $\text{tr} P_a = D_a \phi$. In the next section we will apply this Ansatz to derive the compactification of the \hat{R}^2 term.

Before leaving this review of the dimensional reduction we would like make more explicit the relation of our conventions to the ones in *e.g.* ref. [9]. In that paper the Ansatz is written as

$$\hat{E}^a = e^{\alpha\phi} e^a , \quad \hat{E}^i = e^{\beta\phi} (dy^\mu + A^\mu) \tilde{e}_\mu{}^i , \quad (2.6)$$

where the internal vielbein $\tilde{e}_\mu{}^i$ is an element of $SL(n)$. Furthermore, the parameters α and β are determined to satisfy $\alpha^2 = \frac{n}{2(d-2)(n+d-2)}$ and $\beta = -\frac{d-2}{n}\alpha = -\sqrt{\frac{d-2}{2n(d-2)}}$ in order for the reduction to produce a canonical Einstein–Hilbert term and a properly normalised kinetic term for the scalar ϕ . In fact, using the above Ansatz the coefficient in front of the scalar kinetic terms reads

$$(d-1)(d-2)\alpha^2 + 2n(d-2)\alpha\beta + n(n+1)\beta^2 . \quad (2.7)$$

Since our Ansatz corresponds to $d = 3$, $\alpha = -1$, and $\beta = \frac{1}{n}$ we find the coefficient to be $1 + \frac{1}{n}$. This is consistent with our action in eq. (2.4) above if one extracts the contribution to the scalar kinetic term from the coset term. The choice $\beta = \frac{1}{n}$ is natural, since it keeps intact the $GL(n)$ element that will be a building block of $SL(n+1)$ in the following section. Finally, note that the field strength F^i appearing in eq. (2.4) has an extra ϕ dependence hidden in the internal vielbein.

3. TOROIDAL DIMENSIONAL REDUCTION OF R^2

We now consider adding to the Einstein–Hilbert action terms of higher order in the Riemann tensor . In the present paper, we only treat one such deformation at the time, and think of

it as the next-to-leading term in an infinite expansion in a dimensionful parameter formed from α' or Newton's constant.

At the level of R^2 there is only one possible term, modulo field redefinitions, namely $\hat{R}_{ABCD}\hat{R}^{ABCD}$. At next-to-leading order, field redefinitions give changes in the action containing the lowest order field equations, so any term containing the Ricci tensor can be thrown away without loss of generality. The dimensional reduction (setting $A = (a, i)$ etc) will result in an expression that contains the following kind of terms: the square of R_{abcd} , two F_{ab}^i field strengths contracted to one R_{abcd} , plus F_{ab}^i , P_a^{ij} , and $D_a\phi$ combined into terms with four such fields, or to terms with three or two fields together with one or two covariant derivatives D_a , respectively.

We note at this point that modulo field equations $\hat{R}_{ABCD}\hat{R}^{ABCD}$ is equivalent to the Gauss–Bonnet term $\mathcal{L}_{GB} = \hat{E}(\hat{R}_{ABCD}\hat{R}^{ABCD} - 4\hat{R}_{AB}\hat{R}^{AB} + \hat{R}^2)$. The fact that the integral of this expression, $\int d^D x \mathcal{L}_{GB} \sim \int \varepsilon_{A_1 \dots A_D} \hat{R}^{A_1 A_2} \wedge \hat{R}^{A_3 A_4} \wedge \hat{E}^{A_5} \wedge \dots \wedge \hat{E}^{A_D}$, is a topological invariant in some dimension ($D = 4$) implies that it has no two-point function (the terms quadratic in fields are total derivatives). Perhaps less well-known is that this feature repeats itself at the level of three fields in the scalar sector. This is an effect of the dimensional reduction. It is quite trivial to convince oneself that any three-point coupling $P^2 DP$, modulo the lowest order field equation (representing the freedom of field redefinitions) is a total derivative. However, as we will discuss more later, for R^3 and R^4 terms related to topological invariants in six and eight dimensions, this property holds only for terms containing three and four fields, respectively.

To present the result of the dimensional reduction of $\hat{R}_{ABCD}\hat{R}^{ABCD}$ it is convenient to first note that the splitting of the indices $A = (a, i)$ etc gives

$$\begin{aligned} \hat{R}_{ABCD}\hat{R}^{ABCD} = & \hat{R}_{abcd}\hat{R}^{abcd} + 4\hat{R}_{ibcd}\hat{R}^{ibcd} + 2\hat{R}_{ijcd}\hat{R}^{ijcd} \\ & + 4\hat{R}_{ibkd}\hat{R}^{ibkd} + 4\hat{R}_{ijkd}\hat{R}^{ijkd} + \hat{R}_{ijkl}\hat{R}^{ijkl} . \end{aligned} \quad (3.1)$$

At this point we suppress the dilaton dependence in the higher curvature terms. It should of course be kept for a complete treatment, but will be irrelevant for the considerations in this and the following sections. Formally, this amounts to setting $\phi = 0$, which implies $\text{tr} P = 0$.

We then get

$$\begin{aligned}
 \hat{R}_{abcd}\hat{R}^{abcd} &= R_{abcd}R^{abcd} - \frac{3}{2}R_{abcd}(F^{ab}F^{cd}) \\
 &\quad + \frac{3}{8}[(F^{ab}F^{cd})(F_{ab}F_{cd}) + (F^{ab}F^{cd})(F_{ac}F_{bd})] , \\
 \hat{R}_{ibcd}\hat{R}^{ibcd} &= (D_{[c}F_{d]b}D^cF^{db}) - 2(F_{cd}P_bD^cF^{db}) + (F_{ab}P_cP^cF^{ab}) , \\
 \hat{R}_{ijcd}\hat{R}^{ijcd} &= \frac{1}{8}[(F_a{}^cF_{bc})(F^a{}_dF^{bd}) - (F^{ab}F^{cd})(F_{ac}F_{bd})] + 2(F^{ae}P_{[a}P_{b]}F^b{}_e) \\
 &\quad - 2\text{tr}(P_aP_bP_aP_b) + 2\text{tr}(P^aP_aP^bP_b) , \\
 \hat{R}_{ibkd}\hat{R}^{ibkd} &= \frac{1}{16}(F_a{}^cF_{bc})(F^a{}_dF^{bd}) + \text{tr}(P^aP_aP^bP_b) - \frac{1}{2}(F^a{}_eP_bP_aF^{be}) \\
 &\quad + 2\text{tr}(D_aP_bP^aP^b) - \frac{1}{2}(F^a{}_eD_aP_bF^{be}) + \text{tr}(D_aP_bD^aP^b) , \\
 \hat{R}_{ijkd}\hat{R}^{ijkd} &= \frac{1}{2}(F^a{}_cF^{bc})\text{tr}(P_aP_b) - \frac{1}{2}(F^a{}_eP_bP_aF^{be}) , \\
 \hat{R}_{ijkl}\hat{R}^{ijkl} &= 2\text{tr}(P_aP_b)\text{tr}(P^aP^b) - 2\text{tr}(P_aP_bP^aP^b) .
 \end{aligned} \tag{3.2}$$

All traces and scalar products are over internal indices, all space-time indices are explicit. Two of the above Riemann tensor components depend explicitly, as well as implicitly after integration by parts, on the field equations. After using the lowest order field equations obtained from the reduction of the Einstein–Hilbert term, we find that the expressions for these components become (modulo total derivative terms and including the combinatorial factors above)

$$\begin{aligned}
 4\hat{R}_{ibcd}\hat{R}^{ibcd} &= R^{abcd}(F_{ab}F_{cd}) - 2R^{ab}(F_a{}^cF_{bc}) - \frac{1}{2}(F^{ab}F^{cd})(F_{ab}F_{cd}) \\
 &\quad + 6(F_a{}^cP^bP^aF_{bc}) + 2(F^{ab}P^cP_cF_{ab}) , \\
 4\hat{R}_{ibkd}\hat{R}^{ibkd} &= -4R_{ab}\text{tr}(P^aP^b) + \frac{1}{4}(F_a{}^cF_{bc})(F^a{}_dF^{bd}) + \frac{1}{8}(F^{ab}F^{cd})(F_{ab}F_{cd}) \\
 &\quad - 4\text{tr}(P^aP_aP^bP_b) + 8\text{tr}(P_aP_bP^aP^b) \\
 &\quad - 2(F^a{}_eP_aP_bF^{be}) - 2(F^{ab}P^cP_cF_{ab}) .
 \end{aligned} \tag{3.3}$$

Note that we have not yet implemented the Einstein equation since it will only produce terms with short traces, that is over two P' s, and these will not enter the discussion below. It is for the same reason that we can neglect the dependence on the scalar ϕ in the above formulae. Here we have also made use of the Maurer–Cartan equations and Bianchi identities which in the particular case of R^2 terms implies that no derivatives appear anywhere (it is straightforward to show that this is true also for non-constant ϕ). As we will see in later sections this nice feature will not occur for R^n with $n > 2$.

In $d = 3$ the two-forms F can be dualised to one-forms f , turning the graviphoton degrees of freedom into scalars. Dualisation is performed by adding a term $\int u_\mu dF^\mu$ to the action, thus enforcing the Bianchi identity of F with a Lagrange multiplier, and treating F as

an independent field. Solving the algebraic field equations for F in terms of du and reinserting the solution into the action gives the action in terms of the scalar dual graviphotons u_μ . At the level of the Einstein–Hilbert action, reintroducing the scalar, this procedure gives the Lagrangian

$$\mathcal{L}_{\text{dual}} = e \left[R - \text{tr}(P_a P^a) - \frac{1}{2}(f_a f^a) - D_a \phi D^a \phi \right], \quad (3.4)$$

where the dualised field strength is given by $F^i = e^{-\phi} \star f^i$. It has the Bianchi identity $Df + f \wedge P + f \wedge d\phi = 0$ and equation of motion $D^a f_a - P^a f_a - D^a \phi f_a = 0$, and is obtained from the scalar as $f = e^{-\phi} e^{-1} du$. The dualised scalars fit together with the $GL(n)$ ones parametrisng the internal torus into an element of $SL(n+1)$ as

$$G = \begin{bmatrix} e^{-\phi} & 0 \\ e^{-\phi} u & e \end{bmatrix}, \quad (3.5)$$

which gives the $SL(n+1)$ Maurer–Cartan form

$$\mathcal{P} + \mathcal{Q} = G^{-1} dG = \begin{bmatrix} -d\phi & 0 \\ f = e^{-\phi} e^{-1} du & e^{-1} de \end{bmatrix}. \quad (3.6)$$

The $SL(n+1)$ symmetry of the dimensionally reduced Einstein–Hilbert action is manifested as

$$\mathcal{L}_{\text{dual}} = e \left[R - \text{tr}(\mathcal{P}_a \mathcal{P}^a) \right]. \quad (3.7)$$

We note that, at lowest order, the Lagrange multiplier term contributes to the action (in fact, so that the kinetic term keeps its correct sign after dualisation). When the action contains higher order interaction terms, the equations of motion for F become non-linear, and one will get a non-linear duality relation between F and f . In general one has to be careful about this, but it is straightforward to check that for any next-to-leading term, the non-linearities cancel between the F^2 term and the Lagrange multiplier term. To next-to-leading order, which is all we treat in this paper, the correct dualised version of the higher-curvature term is obtained by direct insertion of the linearly dualised graviphotons.

In view of this it is of course interesting to check if the pure P terms, respecting the manifest $\mathfrak{so}(n)$ symmetry, can combine with the graviphotonic scalars to form the enlarged symmetry $\mathfrak{sl}(n+1)$ also when the R^2 terms are included. To this end we collect the terms of the form $\text{tr}(P_a P_b P^a P^b)$ and $\text{tr}(P_a P^a P_b P^b)$ together with the terms containing F ’s that would mix with them under $\mathfrak{sl}(n+1)$.

The result is

$$2\text{tr}(P_a P_b P^a P^b) + 2(F_{ac} P^b P^a F_b{}^c) \quad (3.8)$$

(*i.e.*, the terms $\text{tr}(P_a P^a P_b P^b)$ cancel out), which becomes, after dualisation of the two-forms F^i to one-forms f^i as discussed above,

$$2\text{tr}(P_a P_b P^a P^b) + 2(f^a P_a P^b f_b) - 2(f^a P_b P^b f_a) . \quad (3.9)$$

This should then be compared to the $SL(n+1)$ -covariant expression $\text{tr}\mathcal{P}^4$. The terms contributing uniquely to this “long trace”, and not to $(\text{tr}\mathcal{P}^2)^2$, are of the types $\text{tr}P^4$ and $(fPPf)$ as above, together with $\partial\phi(fPf)$, with tangent indices placed in all possible ways. With the parametrisation of the $SL(n+1)/SO(n+1)$ coset as above, we get

$$\begin{aligned} \text{tr}(\mathcal{P}^a \mathcal{P}_a \mathcal{P}^b \mathcal{P}_b) &= \text{tr}(P^a P_a P^b P_b) + \frac{1}{2} [(f^a P_b P^a f_b) + (f^a P^b P_b f_a)] \\ &\quad + \frac{1}{16} [(f^a f_a)(f^b f_b) + (f^a f^b)(f_a f_b)] + \pi^a (f_a P^b f_b) \\ &\quad + \frac{1}{2} [\pi^a \pi_a (f^b f_b) + \pi^a \pi^b (f_a f_b)] + \pi^a \pi_a \pi^b \pi_b , \\ \text{tr}(\mathcal{P}^a \mathcal{P}^b \mathcal{P}_a \mathcal{P}_b) &= \text{tr}(P^a P^b P_a P_b) + (f^a P^b P_a f_b) \\ &\quad + \frac{1}{8} (f^a f^b)(f_a f_b) + \pi^a (f^b P_a f_b) + \pi^a \pi^b (f_a f_b) + \pi^a \pi_a \pi^b \pi_b , \end{aligned} \quad (3.10)$$

where $\pi = -d\phi$ is the upper left corner component of \mathcal{P} . It seems hard to reconcile eq. (3.9) with a possible $\mathfrak{sl}(n+1)$. In fact, the coefficients of the two terms are dictated by the $\text{tr}P^4$ terms. Of the three structures $(fPPf)$ consistent with $SL(n)$, only two linear combinations are allowed by $SL(n+1)$. The terms from dimensional reduction in eq. (3.9) are not the ones required by eq. (3.10).

In the above calculation, the volume factor e^ϕ of the internal torus has been omitted (set to 1). After dualisation, any term from \hat{R}^p carries an overall factor $e^{2(p-1)\phi}$. This factor tells us that the terms obtained by dimensional reduction cannot be $SL(n+1)$ -invariant, since ϕ is one of the scalars parametrising the coset $SL(n+1)/SO(n+1)$. Neither is this expected from string theory or M-theory, since quantum corrections break the global symmetry group to a discrete version. The terms obtained from the reduction will not be the whole answer, but its large volume limit. The torus volume factor may be obtained as the large volume limit of an automorphic form. As we will see later, the observation that the tensor structure does not match with $SL(n+1)$ covariance means that scalar ($SO(n+1)$ -invariant) automorphic forms (*i.e.*, functions) do not suffice, and calls for the introduction of automorphic forms transforming under $SO(n+1)$. Similar conclusions are reached in ref. [10] based on an investigation of the root and weight structure of the scalar prefactors.

At this point, we could of course extend the investigation to other terms by including $d\phi$ and considering also “short” traces. However, as we already have demonstrated the need for

transforming automorphic forms, we will now show how any term obtained in the reduction can be matched to such constructions.

4. TRANSFORMING AUTOMORPHIC FORMS

Previous work by Green et al [19] (see also [11]) indicates how the apparent contradiction found in the previous section should be resolved. In fact, as we will see in later sections, there are also terms arising in the compactification of R^4 from $D = 11$ to $d = 3$ that are not immediately compatible with the $SL(9)$ subgroup of $E_{8(8)}$. We suggest that the proper interpretation of these results is that they should be viewed as the large volume limit of an $SL(9, \mathbb{Z})$ -invariant constructed from transforming automorphic forms and non-scalar products of the fields in question. This turns out to hold for the R^2 terms of the previous section on reduction from any D to $d = 3$. Of course, consistency with decompactification requires that the automorphic form, in the large volume limit, does not diverge and has as its only remnant after decompactification the very term that was used as starting point for the compactification.

Appendix A describes the construction of automorphic forms, scalar as well as transforming ones. (For a partly overlapping discussion see the Appendix of ref. [10].) We now apply this construction to the quartic terms of the previous section, although it will be obvious that the treatment is general. For any irreducible $SO(n+1)$ representation r contained in the symmetric product of four symmetric traceless tensors, we can form the combination $\psi_{IJ, KL, MN, PQ}^{(r)} \mathcal{P}^{aIJ} \mathcal{P}_a^{KL} \mathcal{P}^{bMN} \mathcal{P}_b^{PQ}$, where ψ is an automorphic form transforming in the representation r . The symmetric product of four symmetric traceless $SO(n+1)$ tensors contains 23 irreducible representations for any $n \geq 8$, and this is then the number of $SL(n+1, \mathbb{Z})$ -invariant terms we can write down starting from the symmetric traceless representation*. This is however true only when all the indices on \mathcal{P} 's are contracted with indices on an automorphic form constructed as in appendix A. The actual number is larger, since nothing prevents us from taking products of such automorphic forms and invariant tensors without symmetrising all indices—there is no *a priori* reason to symmetrise in \mathcal{P} 's with different space-time indices.

In the case of an SL group, it is preferable to build automorphic forms from the fundamental representation (although this option does not exist if we want *e.g.* $SL(9)$ as a

* The “weight” of each automorphic form, as defined in appendix A, is fixed by the overall volume factor. We ignore ambiguities from products of automorphic forms, where only the sum of weights will be determined, as well as from the use of different Casimirs in the sum defining the automorphic form. Terms differing in these respects are indistinguishable in the large volume limit.

subgroup of $E_{8(8)}$). By using the automorphic forms built from the fundamental representation, we have seen in Appendix A that the only surviving part in the large volume limit is the one with all indices equal to 0 (the first component in our $SL(n+1)$ matrices) [10]. The part of \hat{R}^2 containing P^4 comes from an $SO(n+1)$ scalar automorphic form. Since $\mathcal{P}_{0i} = \frac{1}{2}f_i$ and $\mathcal{P}_{00} = \pi$, we can always choose to insert an even number of zeros in the positions we like, and thereby arrange for products of transforming automorphic forms and $SO(n+1)$ -invariant tensors to have a large volume limit reproducing any of the $SO(n)$ -invariant terms occurring in the reduction. The matching can be made recursively, in increasing number of 0 indices. We take the long trace as example. The terms with scalar automorphic forms are determined from the $\text{tr}P^4$ terms to be proportional to $\psi^{(1)}\text{tr}(\mathcal{P}^a\mathcal{P}^b\mathcal{P}_a\mathcal{P}_b)$, with the notation of Appendix A. Subtracting its large volume limit from the actual result of the reduction, given in part by eq. (3.9), there is a remainder proportional to $(f^a P_a P^b f_b) - (f^a P^b P_b f_a) - (f^a P^b P_a f_b)$. This implies a term proportional to $\psi_{IJ}^{(2,1)}(\mathcal{P}^a\mathcal{P}_a\mathcal{P}^b\mathcal{P}_b - \mathcal{P}^a\mathcal{P}^b\mathcal{P}_b\mathcal{P}_a - \mathcal{P}^a\mathcal{P}^b\mathcal{P}_a\mathcal{P}_b)^{IJ}$. With a more complete expansion of the reduced curvature term, it can always be matched to the large volume limit of an expression in terms of \mathcal{P} 's and automorphic forms.

It should of course be checked that automorphic forms exist that give the correct power of the torus volume factor obtained from the reduction. A term \hat{R}^p gives an overall factor $e^{2(p-1)\phi}$. Suppose we try to obtain some corresponding terms with a product of M automorphic forms, each with $2l_k$ fundamental indices and weight w_k , $k = 1, \dots, M$. Convergence of the sum defining the automorphic forms demands $2(w_k - l_k) > 1$, and the large volume limit will yield a dilaton dependence $\exp(\sum_{k=1}^M 2(w_k - l_k)\phi)$. Matching with the reduction gives $p-1 = \sum_{k=1}^M (w_k - l_k)$ and thus $M < 2(p-1)$. The \hat{R}^2 term gives room for one automorphic form, which is exactly what we need.

It is not clear what to expect for this kind of symmetry enhancement to $SL(n+1)$ in the context of \hat{R}^2 terms. There are *e.g.* no known examples from string/M-theory that make use of such a step, so the \hat{R}^2 term should probably be seen as a toy model to set the framework for the higher-curvature terms. (The heterotic string has an \hat{R}^2 term, but the symmetry enhancement is to $SO(n,n)$; this case is treated in ref. [21].) The situation is different for the \hat{R}^4 terms, which are the first higher derivative terms to arise in the maximally supersymmetric string theories in 10 dimensions, as well as in 11-dimensional M-theory. When compactified to three dimensions all degrees of freedom are collected into a coset based on $E_{8(8)}$ when starting from a two-derivative action. We now proceed to discuss \hat{R}^3 and \hat{R}^4 terms in this context with the goal of understanding the role of $SL(n+1)$, and to develop methods that might eventually be useful in dealing with the more complicated case of $E_{8(8)}$.

Automorphic forms of $SL(2, \mathbb{Z})$ transforming under $U(1)$ have been encountered in loop calculations with external fermions in string theory compactified on a circle to $d = 9$ [19]. We expect that the appearance of transforming automorphic forms is generic in a situation

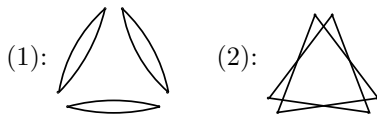
where the external fields of the diagram transforms under K . In the specific case in ref. [19], the contribution was shown to disappear in the M-theoretic large volume limit. We note that this limit is a quite different one in terms of the SL group involved than in our case. We deal with a larger symmetry $SL(n+1)$ appearing after dualisation of the graviphotons, and blow up a certain parameter, that, had our $SL(n+1)$ element been a vielbein on T^{n+1} , would have corresponded to shrinking one direction and consequently blowing up the other n directions. In the $SL(2)$ case, the large volume limit has nothing to do with the $SL(2)$ element parametrising the shape of T^2 , but with the determinant of a $GL(2)$ element blowing up. We have shown how to match combinations of \mathcal{P} 's and automorphic forms that are designed to survive in the large volume limit. It would be interesting to compare such a construction (not for terms corresponding to \hat{R}^2 , but presumably to \hat{R}^4) to actual loop calculations.

5. THE CASE OF R^3

Our main concern in the rest of the paper is the investigation of the \hat{R}^4 terms which are part of the first non-trivial correction in M-theory and type II string theory. Before doing that we would however like to emphasize some aspects of \hat{R}^3 terms. The \hat{R}^2 terms of section 3 were a testing ground for the ideas but turned out to have some special non-generic features, such as the effective vanishing of all terms with second derivatives on scalar fields. As we will see below this feature is not found for \hat{R}^3 and higher terms. Here we also take the opportunity to introduce some diagrammatic methods that will be tremendously helpful in keeping track of index structures of increasing complexity as we go to higher powers of the Riemann tensor.

Again, the \hat{R}^3 terms are seen as a next-to-leading order correction to the Einstein–Hilbert action (*i.e.*, there are no \hat{R}^2 terms). Any term which contains lowest order field equation can be removed by a field redefinition, so we leave them out from the start. We thus want to list all possible terms where indices are contracted between different Riemann tensors. We represent each contracted index by a line, and each Riemann tensor by the endpoints of four such lines. The lines whose endpoints meet represent an antisymmetric pair of indices. The sign is fixed by letting the indices, as they sit on \hat{R} , run in the clockwise direction in the diagram. The only structure not accounted for is $\hat{R}_{A[BCD]} = 0$, which has a simple diagrammatic expression.

A basis for the two inequivalent R^3 terms can be taken as



thus representing $\hat{R}_{AB}{}^{CD}\hat{R}_{DC}{}^{EF}\hat{R}_{FE}{}^{BA}$ and $\hat{R}_A{}^B{}_D{}^E\hat{R}_B{}^C{}_E{}^F\hat{R}_C{}^A{}_F{}^D$, respectively. One may also consider the contraction \triangletriangle , but it is related to the ones in the basis, using the Bianchi identity $\hat{R}_{A[BCD]} = 0$, as $\triangletriangle = \frac{1}{4}(1) + (2)$.

At this level, there is one obvious combination that does not give any three-point couplings. This is the “Gauss–Bonnet” term,

$$\begin{aligned} \varepsilon\varepsilon\hat{R}^3 &= \varepsilon^{A_1\dots A_D}\varepsilon_{B_1\dots B_D}\hat{R}_{A_1A_2}{}^{B_1B_2}\hat{R}_{A_3A_4}{}^{B_3B_4}\hat{R}_{A_5A_6}{}^{B_5B_6}\delta_{A_7\dots A_D}^{B_7\dots B_D} \\ &= 32(D-6)!\{(1) + 2(2)\} \end{aligned} \quad (5.1)$$

which is topological in $D = 6$ and lacks three-point couplings in any dimension. The general form of the scalar terms will be $(D\mathcal{P})^3 + \mathcal{P}^2(D\mathcal{P})^2 + \mathcal{P}^4D\mathcal{P} + \mathcal{P}^6$, where “ \mathcal{P} ” denotes any of P , f and $\partial\phi$, but we are guaranteed that the first term vanish for this specific combination.

To see this explicitly, and to derive further properties relying on the dimensional reduction, we concentrate on the pure P terms (note that this truncation is consistent and implies $\text{tr}P = 0$). They are extracted in the Riemann tensor derived in section 2:

$$\begin{aligned} \hat{R}_{ab}{}^{cd} &= -4\delta_{[a}^{[c}\text{tr}(P_{b]}P^{d]}) + \delta_{[a}^c\delta_{b]}^d\text{tr}(P_eP^e) , \\ \hat{R}_{ab}{}^{cl} &= 0 , \\ \hat{R}_{ab}{}^{kl} &= -2(P_{[a}P_{b]})^{kl} , \\ \hat{R}_{aj}{}^{cl} &= -(D_aP^c)_j{}^l - (P_aP^c)_j{}^l , \\ \hat{R}_{aj}{}^{kl} &= 0 , \\ \hat{R}_{ij}{}^{kl} &= -2(P_e)_{[i}{}^k(P^e)_{j]}{}^l . \end{aligned} \quad (5.2)$$

Notice that Einstein’s equations in three dimensions have been used to obtain the specific form of $\hat{R}_{ab}{}^{cd}$.

The two independent cubic contractions of the Riemann tensor components above become after compactification, and keeping only terms which give pure P contributions*,

$$\begin{aligned} &\hat{R}_{AB}{}^{CD}\hat{R}_{DC}{}^{EF}\hat{R}_{FE}{}^{BA} \\ &= R_{ij}{}^{kl}R_{lk}{}^{mn}R_{nm}{}^{ji} + 3R_{ab}{}^{kl}R_{lk}{}^{mn}R_{nm}{}^{ba} + 8R_{aj}{}^{cl}R_{lc}{}^{en}R_{ne}{}^{ja} \\ &+ R_{ab}{}^{cd}R_{dc}{}^{ef}R_{fe}{}^{ba} + 3R_{ab}{}^{cd}R_{dc}{}^{ij}R_{ji}{}^{ba} \end{aligned} \quad (5.3)$$

* The combinatorial factors are easily read off from the diagrams. Splitting of the indices in two classes, space-time and internal, corresponds to coloring the lines in the diagrams with two colors. The factors are given by the number of ways this can be done.

and

$$\begin{aligned} & \hat{R}_A{}^B{}_D{}^E \hat{R}_B{}^C{}_E{}^F \hat{R}_C{}^A{}_F{}^D \\ &= R_i{}^j{}_l{}^m R_j{}^k{}_m{}^n R_k{}^i{}_n{}^l + 3R_a{}^j{}_d{}^m R_j{}^k{}_m{}^n R_k{}^a{}_n{}^d + 2R_i{}^j{}_d{}^e R_j{}^k{}_e{}^f R_k{}^i{}_f{}^d \\ &+ 6R_i{}^b{}_d{}^m R_b{}^k{}_m{}^f R_k{}^i{}_f{}^d + R_a{}^b{}_e{}^f R_b{}^c{}_f{}^g R_c{}^d{}_g{}^e + 3R_a{}^b{}_c{}^d R_b{}^i{}_d{}^j R_i{}^a{}_j{}^c . \end{aligned} \quad (5.4)$$

We now insert the P -dependent terms from above. For the purposes here it is sufficient to collect only the $(DP)^3$ and $P^2(DP)^2$ terms, while remembering that $\text{tr}P = 0$.

For $(DP)^3$, which gets contribution entirely from $(1)^\dagger$, it is straightforward to show that it is a total derivative

$$(DP)^3 = \text{tr}(S_a{}^b S_b{}^c S_c{}^a) = D_c [\text{tr}(P^a S_a{}^b S_b{}^c) - \frac{1}{2} \text{tr}(P^c S_a{}^b S_b{}^a)] \quad (5.5)$$

(as always, modulo lowest order equations of motion), where $S_{ab} = D_a P_b$. The tensor $(S_{ab})_{ij}$ is symmetric in both (ab) and (ij) , and $S_a{}^a$ is the kinetic term in the equation of motion for P_a . The fact that, modulo equations of motion, the $(DP)^3$ term is a total derivative is expected for the highest derivative term in a Gauss–Bonnet combination of any order, but we see that after dimensional reduction the scalar three-point couplings vanish for any \hat{R}^3 term.

Doing a similar analysis for the $(DP)^2 P^2$ terms, there are 10 algebraically independent structures

$$\begin{aligned} (i) &= \text{tr}(S_a{}^b S_b{}^a P_c P^c) , \\ (ii) &= \text{tr}(S_a{}^b S_b{}^c P_c P^a) , \\ (iii) &= \text{tr}(S_a{}^b S_b{}^c P^a P_c) , \\ (iv) &= \text{tr}(S_a{}^b P_c S_b{}^a P^c) , \\ (v) &= \text{tr}(S_a{}^b P_b S^a{}_c P^c) , \\ (vi) &= \text{tr}(S_a{}^b S_b{}^a) \text{tr}(P_c P^c) , \\ (vii) &= \text{tr}(S_a{}^b S_b{}^c) \text{tr}(P_c P^a) , \\ (viii) &= \text{tr}(S_a{}^b P_c) \text{tr}(S_b{}^a P^c) , \\ (ix) &= \text{tr}(S_a{}^b P_c) \text{tr}(S_c{}^a P_b) , \\ (x) &= \text{tr}(S_a{}^b P_b) \text{tr}(S_c{}^a P^c) . \end{aligned} \quad (5.6)$$

Since $(i)–(v)$ will not mix with $(vi)–(x)$, we will consider the two groups separately. For the single trace terms, neglecting the equations of motion, the combination $x_1(i) + 2x_2(ii) - (x_2 - 2x_3)(iii) + x_3(iv) + (2x_1 - x_2)(v)$ is a total derivative for arbitrary values of $\{x_*\}$.

[†] This term comes only from $R_{aj}{}^{cl}$, which means that it gets contributions only from colored diagrams with alternating color on all cycles. A diagram containing a cycle with an odd number of lines can not contribute.

Correspondingly a total derivative consisting of the double trace terms must be written as $y_1(vi) + (y_2 + y_3)(vii) + y_2(viii) + (y_2 - 2y_3)(ix) + (2y_1 + y_3)(x)$ for arbitrary values of $\{y_*\}$. Extracting the pure $(DP)^2 P^2$ terms from (5.3) and (5.4) we find that

$$(1) + z(2) = 6(4 - z)(ii) + 6z(iii) + 3z(iv) + z \left[-\frac{3}{2}(vi) + 6(vii) - 3(viii) \right] , \quad (5.7)$$

with an arbitrary parameter z . For the single trace terms in eq. (5.7) $z = 4$ is the only choice where they can form a total derivative, this corresponds to the case $x_1 = x_2 = 0$, $x_3 = 12$, (which is not the Gauss–Bonnet combination from eq. (5.1)). For the double trace terms in eq. (5.7), however, no choice of z can make them a total derivative. We have thus shown that the $(DP)^2 P^2$ cannot vanish by partial integrations. Unlike in the \hat{R}^2 terms, derivatives of Maurer–Cartan forms necessarily appear.

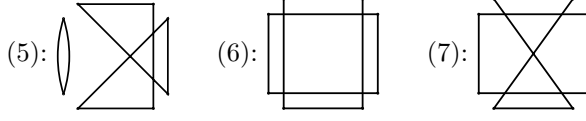
A more complete treatment should include also the other fields in \mathcal{P} . One should also continue with terms of the types $\mathcal{P}^4 D\mathcal{P}$ and \mathcal{P}^6 . This would imply quite some work which we do not find motivated for \hat{R}^3 . In order to access the complete expressions, care has to be taken when using partial integrations, since terms which a certain number of derivatives contributes to terms with fewer derivatives via equations of motion, Bianchi identities and curvatures (R and F_Q).

6. R^4 TERMS

In this section we start the analysis of the \hat{R}^4 terms by presenting the content of $t_8 t_8 \hat{R}^4$ and $\varepsilon \varepsilon \hat{R}^4$ in terms of an explicitly given basis of seven elements. That this basis is seven-dimensional is well-known [22]. We then concentrate on the terms that after the dimensional reduction contain only the coset variable P_a^{ij} . These are of the types $(DP)^4$, $P^2(DP)^3$, $P^4(DP)^2$, $P^6(DP)$, and P^8 . A test of the possible rôle of the octic invariant of $E_{8(8)}$ derived in ref. [23] is spelt out (for details see Appendix B). This would involve the P^8 terms and be rather lengthy. For that reason we turn in section to the much simpler terms $(DP)^4$ from which we are able to draw the conclusions we are looking for.

Using the same diagrammatic notation as in the previous section, a basis for the seven \hat{R}^4 terms can be taken as

$$(1): \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad (2): \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad (3): \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad (4): \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ \text{---} \end{array}$$



A contraction that also occurs naturally (*e.g.* in $\varepsilon\varepsilon R^4$) is $\boxed{\times}$, and it can be related to the others by using $\hat{R}_{[ABC]D} = 0$ as follows: Cycling on $\boxed{\times}$ gives $\boxed{\times} = \boxed{\times} - \frac{1}{2}\boxed{\times}$. Cycling on (5) = $\boxed{\times}$ gives $\boxed{\times} = \boxed{\times} + \frac{1}{2}\boxed{\times}$, and on (7) = $\boxed{\times}$ gives $\boxed{\times} = \boxed{\times} + \frac{1}{2}\boxed{\times}$. Eliminating the diagrams not present in the basis, $\boxed{\times}$ and $\boxed{\times}$, gives the relation $\boxed{\times} = \frac{1}{4}(4) - (5) + (7)$.

In $D = 10$ and 11, the structures

$$\varepsilon\varepsilon\hat{R}^4 = \varepsilon^{A_1\dots A_D}\varepsilon_{B_1\dots B_D}\hat{R}_{A_1A_2}{}^{B_1B_2}\hat{R}_{A_3A_4}{}^{B_3B_4}\hat{R}_{A_5A_6}{}^{B_5B_6}\hat{R}_{A_7A_8}{}^{B_7B_8}\delta_{A_9\dots A_D}^{B_9\dots B_D} \quad (6.1)$$

(the “Gauss–Bonnet term”) and $t_8 t_8 \hat{R}^4$ are of special interest, since they, or combinations of them, are dictated by string theory calculations and by supersymmetry; see for instance the explicit evaluation in Appendix B2 of ref. [12] of the appropriate superspace term given in ref. [24]. The invariant tensor $t_8^{A_1A_2,A_3A_4,A_5A_6,A_7A_8}$ is defined to be antisymmetric in the indices composing the pairs and symmetric in the four pairs. When contracted with the an antisymmetric matrix M , it is defined to give

$$t_8^{A_1A_2,A_3A_4,A_5A_6,A_7A_8}M_{A_1A_2}M_{A_3A_4}M_{A_5A_6}M_{A_7A_8} = 24\text{tr}M^4 - 6(\text{tr}M^2)^2. \quad (6.2)$$

In $t_8 t_8 \hat{R}^4$, the indices are contracted according to

$$t_8 t_8 \hat{R}^4 = t_8^{A_1A_2,A_3A_4,A_5A_6,A_7A_8} t_8^{B_1B_2,B_3B_4,B_5B_6,B_7B_8} \times \hat{R}_{A_1A_2}{}^{B_1B_2} \hat{R}_{A_3A_4}{}^{B_3B_4} \hat{R}_{A_5A_6}{}^{B_5B_6} \hat{R}_{A_7A_8}{}^{B_7B_8}. \quad (6.3)$$

Direct evaluation gives, in D dimensions,

$$\begin{aligned} \frac{1}{12} t_8 t_8 \hat{R}^4 &= 2(1) + (2) - 16(3) - 8(4) && + 16(6) + 32(7) \\ -\frac{1}{48(D-8)!} \varepsilon\varepsilon\hat{R}^4 &= 2(1) + (2) - 16(3) && + 32(5) + 16(6) && - 32\boxed{\times} \end{aligned} \quad (6.4)$$

or, with $\boxed{\times}$ expressed in the basis as above,

$$\begin{aligned} \frac{1}{12} t_8 t_8 \hat{R}^4 &= 2(1) + (2) - 16(3) - 8(4) && + 16(6) + 32(7) \\ -\frac{1}{48(D-8)!} \varepsilon\varepsilon\hat{R}^4 &= 2(1) + (2) - 16(3) - 8(4) + 64(5) + 16(6) - 32(7) \end{aligned} \quad (6.5)$$

These expressions agree with the ones in *e.g.* ref. [17], where the basis $\{A_1 = (2), A_2 = (3), A_3 = (1), A_4 = (4), A_5 = -\boxtimes = \frac{1}{2}(4) - (5), A_6 = (6), A_7 = \boxtimes = \frac{1}{4}(4) - (5) + (7)\}$ is used.

The P^8 terms obtained when compactifying from eleven dimensions to three will of course form a scalar of $SO(8)$. Assuming that these terms combine to a scalar also of the $Spin(16)/\mathbb{Z}_2$ that is associated with coset $E_{8(8)}/(Spin(16)/\mathbb{Z}_2)$ arising in the two-derivative sector of M-theory, the only invariant possible (apart from the fourth power of the quadratic one) would be the octic invariant constructed in ref. [23]. As explained in Appendix B, when reducing this to an invariant of $SO(9)$ one finds a certain polynomial in the $SL(9)/SO(9)$ coset element that if valid puts severe restrictions on the structure of the P^8 terms. However, checking this is lengthy and instead we turn to the $(DP)^4$ terms where, as we will see below, some qualitative results we are looking for can be obtained with much less effort.

Thus, we now concentrate on the four-point couplings, which consequently have four derivatives. Assume, for the moment, that $E_{8(8)}(\mathbb{Z})$ invariance were to be achieved with a scalar automorphic form. Since E_8 has no invariant of order four other than the square of the quadratic Casimir (and thus the only $\mathfrak{so}(16)$ invariant quartic in spinors is the square of the quadratic one), we would get the restriction that any trace $\text{tr}(DP)^4$ has to vanish, since this $\mathfrak{so}(8)$ invariant cannot be lifted via $\mathfrak{so}(9)$ to $\mathfrak{so}(16)$.

Using the Riemann tensor with only P terms (see the previous section), it is not very difficult to derive the $(DP)^4$ -terms from the diagrams (1)-(7). Since we only want contributions with the components R_{abij} , one gets one contribution from each coloring with two colors (for space-time and internal indices) of the graphs, where the two colors alternate on every cycle. It follows directly that any diagram with a cycle of odd length does not contribute. There are none in the basis, but in the process of cycling above we encountered the contraction $\boxtimes = (5) - \frac{1}{2}(4)$ that then does not contribute to $(DP)^4$.

There are 8 algebraically independent structures containing $(DP)^4$. We enumerate them as

$$\begin{aligned}
 (i) &= \text{tr}(S_{ab}S^{ab}S_{cd}S^{cd}) , \\
 (ii) &= \text{tr}(S_{ab}S_{cd}S^{ab}S^{cd}) , \\
 (iii) &= \text{tr}(S_{ab}S^{bc}S_{cd}S^{da}) , \\
 (iv) &= \text{tr}(S_{ab}S_{cd}S^{ac}S^{bd}) , \\
 (v) &= \text{tr}(S_{ab}S^{ab})\text{tr}(S_{cd}S^{cd}) , \\
 (vi) &= \text{tr}(S_{ab}S^{cd})\text{tr}(S_{ab}S^{cd}) , \\
 (vii) &= \text{tr}(S_{ac}S^{bc})\text{tr}(S^{ad}S_{bd}) , \\
 (viii) &= \text{tr}(S_{ab}S_{cd})\text{tr}(S^{ac}S^{bd}) ,
 \end{aligned} \tag{6.6}$$

where $S_{ab} = D_a P_b$. One also has to take total derivatives into account. This can be done by writing out all possible terms $(PS^3)_a$ (there are 12) and take the divergence. As long as we only consider $(DP)^4$, we let $S^a_a \rightarrow 0$ and $S_{[ab]} \rightarrow 0$. It turns out that only two combinations of these do not produce terms $P(DP)^2 D^2 P$ (the second derivative of P can again be considered as symmetric and traceless), and they lead to the combinations $(i) + \frac{1}{2}(ii) - (iii) - 2(iv)$ and $\frac{1}{2}(v) + (vi) - 2(vii) - (viii)$ being total derivatives. (These in fact arise from $\text{tr}(P_{[a} S_b^b S_c^c S_d^d)$ and $\text{tr}(P_{[a} S_b^b) \text{tr}(S_c^c S_d^d)$, where the antisymmetry, by the Bianchi identity, prevents $P(DP)^2 D^2 P$ from arising. The counting also holds for reduction to $d = 3$, but with the combinations being total derivatives in higher dimensions now being identically zero.)

Evaluating the contributions to the 4-point couplings from the terms (1), ..., (7) then gives

$$\begin{aligned}
(1) &\longrightarrow 16(iii) , \\
(2) &\longrightarrow 16(v) , \\
(3) &\longrightarrow 4(i) + 4(vii) , \\
(4) &\longrightarrow 8(iv) , \\
(5) &\longrightarrow 4(iv) , \\
(6) &\longrightarrow 2(iii) + 2(vi) , \\
(7) &\longrightarrow (ii) + 2(iv) + (viii) .
\end{aligned} \tag{6.7}$$

Demanding that the contribution vanishes, modulo total derivatives, tells that the R^4 term is proportional to $2(1) + (2) - 16(3) + x(4) + (48 - 2x)(5) + 16(6) - 32(7)$ for some number x . $\varepsilon \varepsilon \hat{R}^4$ (of course) passes the test, but $t_8 t_8 \hat{R}^4$ does not. The combination $(4) - 2(5)$ does, as seen above. In this calculation, $t_8 t_8 \hat{R}^4$ does not even contribute with $(\text{tr} S^2)^2$ terms only, as would be demanded from E_8 invariance. (The condition that the long contractions vanish can be expressed as conditions on the coefficients in front of (5), (6) and (7), given the ones in front of (1)-(4). The latter are identical in $t_8 t_8 \hat{R}^4$ and $\varepsilon \varepsilon \hat{R}^4$.) The “difference” between $t_8 t_8 \hat{R}^4$ and $\varepsilon \varepsilon \hat{R}^4$ (with the normalisations above) is another very simple expression, proportional to $(7) - (5)$ or to $\square - \frac{1}{4}(4)$, whose contribution to the long contractions is $(ii) - 2(iv) \neq 0$. In conclusion, if the term $t_8 t_8 \hat{R}^4$ is present, there are four-point couplings not only in the gravity sector but also in the scalar sector. The term $t_8 t_8 \hat{R}^4$ can not be obtained without transforming E_8 automorphic forms.

We thus find a contradiction with E_8 unless transforming automorphic forms are introduced. The fact that E_8 does not have primitive fourth order invariant means that the $SL(8)$ invariant $D^4 P^4$ terms derived here must come from an E_8 term which is a double trace. Since we find non-zero single trace terms it means that the enhanced symmetries do not generalise to higher derivative terms obtained through compactification as described here with scalar automorphic forms.

Given that the number of automorphic forms of E_8 is smaller than that of $SL(9)$, for the same number of $\mathfrak{so}(16)$ spinors or $\mathfrak{so}(9)$ symmetric traceless tensors (see appendix A), it seems reasonable to believe that E_8 puts some constraints on the possible terms obtained by reduction of pure gravity. Checking this would require more concrete knowledge of E_8 automorphic forms and their large volume limit, as well as (presumably) a much more complete expansion of the seven R^4 terms. It is not at all clear to what degree E_8 will single out some specific combination of these.

Performing a loop calculation with external scalars analogous to the ones in refs. [25,19] would give information on what kind automorphic functions actually appear in an M-theory context (although such a calculation leaves out non-perturbative information from winding membranes and five-branes).

APPENDIX A: AUTOMORPHIC FORMS

Consider an element $g \in G$, where G is a Lie group. In the context of the supergravities (or sigma models) we are considering, g represents the scalar degrees of freedom. These belong to a coset G/K , where K is a subgroup of G . In all cases under consideration, G has the split (maximally non-compact) real form, and K is the maximal compact subgroup of G . The coset is realised by gauging the local right action of K , $g \rightarrow gk$, $k \in K$. This still leaves room for a global left action of G on g , $g \rightarrow \gamma g$, $\gamma \in G$. These global G transformations are however symmetries only of the undeformed supergravities or sigma models, and are broken by quantum effect in string theory. Higher-derivative corrections to effective actions in string theory are expected to break G to a discrete duality subgroup $G(\mathbb{Z})$, and the correct moduli space for the scalars is not G/K but $G(\mathbb{Z}) \backslash G/K$.

The definition of $G(\mathbb{Z})$ has to be clear, of course. If G is a classical matrix group, it can be defined as the group of elements in G with integer entries in the fundamental representation. For exceptional groups, care has to be taken to choose the relevant discrete subgroup. Ref. [26] gives a definition of $G(\mathbb{Z})$ in terms of generators of the Lie algebra \mathfrak{g} of G in the Chevalley basis (see also ref. [2]). In the following it will be understood that $G(\mathbb{Z})$ is the discrete duality group relevant to M-theory compactifications, although the construction in principle holds also for other discrete subgroups of G .

The general method for building automorphic forms [27,28,25,11,10] is to combine g with some element in the discrete group (or a representation of it) so that the resulting entity only transforms under K , in the sense defined below. The invariance under $G(\mathbb{Z})$ is then obtained by summation over $G(\mathbb{Z})$ (or some representation). Let $g \in G$ and $\mu \in G(\mathbb{Z})$, with the transformation rules under $G(\mathbb{Z}) \times K$ with group element $\gamma \otimes k$: $g \rightarrow \gamma g k$, $\mu \rightarrow \gamma \mu \gamma^{-1}$. If

one forms $g^{-1}\mu g$, it transforms as $g^{-1}\mu g \rightarrow k^{-1}(g^{-1}\mu g)k$, *i.e.*, only under K . One may then K -covariantly project $g^{-1}\mu g$ on the representation “ $\mathfrak{g}/\mathfrak{k}$ ”, *i.e.*, the complement to \mathfrak{k} in the Lie algebra \mathfrak{g} , which forms a representation of K .^{*} We denote the obtained building block $\Gamma = \Pi_{\mathfrak{g}/\mathfrak{k}}(g^{-1}\mu g)$. When using tensor notation, we write Γ_α , inspired by the $\mathfrak{so}(16)$ spinor index carried by the tangent space to $E_8(8)/(Spin(16)/\mathbb{Z}_2)$.

Let us start with the simplest kind of automorphic forms, the scalar ones. In order for the function not to transform under K , we need to form scalars from a number of Γ 's. This is straightforward—the algebraically independent polynomial invariants have the same number and degree of homogeneity as the Casimir operators of \mathfrak{g} . In fact, as observed in ref. [23], they are simply the restrictions of the Casimir operators to $\mathfrak{g}/\mathfrak{k}$. Let us denote them $C_i(\Gamma)$, $i = 1, \dots, r$, r being the rank of \mathfrak{g} . Finally, in order to achieve invariance under $G(\mathbb{Z})$, one has to form some function of the C_i 's and sum over the discrete group element μ . The function should be conveniently formed so that the sum converges, *e.g.* a power function. For some “weight” w , we thus define

$$\phi^{(i,w)}(g) = \sum_{\mu \in G(\mathbb{Z})} [C_i(\Gamma(\mu, g))]^{-w} . \quad (A.1)$$

This automorphic function is clearly a function on the double coset $G(\mathbb{Z}) \backslash G/K$.

The construction above is entirely based on Γ , which is obtained as (a projection of) the action of g by conjugation on a discrete group element μ . Alternatively, one may start from some representation. Especially, when G is a classical matrix group, it is simpler to let m lie in the fundamental module (a row vector with integer entries) [28]. Consider the case $G = SL(n)$ with $K = SO(n)$. We form mg , which if m transforms as $m \rightarrow m\gamma^{-1}$ transforms under $G(\mathbb{Z}) \times K$ as $mg \rightarrow (mg)k$. Then, mg is taken as a building block, and one forms the invariant $|mg|^2 = (mg)(mg)^t$. The automorphic function is

$$\psi^{(w)}(g) = \sum_{\mathbb{Z}^n \setminus 0} |mg|^{-2w} . \quad (A.2)$$

This construction has the advantage that the summation is easier to perform than the one over the discrete group, but it is not available for exceptional groups G . Choosing other modules yields algebraically independent automorphic functions, as long as these modules are formed by anti-symmetrisation from the fundamental one. One gets again a number of functions equating the rank.

^{*} This projection may be performed by letting g and μ be represented as matrices in any faithful representation of G , the result of course being independent of the choice of representation.

We expect that the summation in the defining equation (A.1), which is over a single orbit of the discrete group, namely the group itself, can be lifted to the summation over a lattice, quite analogously to how the summation in eq. (A.2) can be decomposed into an infinite number of orbits. Such a lattice summation might make even automorphic forms of exceptional groups reasonable to handle. Eq. (A.1), with the replacement of the discrete group by a lattice, is well suited for the bosonic degrees of freedom of the sigma model obtained by dimensional reduction, since the object Γ carries the same index structure as P . When it comes to fermions, these transform under another representation which is (an enlargement of) a spinor representation of $\mathfrak{so}(n)$, and it seems natural to consider spinorial automorphic forms.

One attractive feature of invariant automorphic forms, automorphic functions, is that their structure and number closely reflect the properties of the Lie algebra \mathfrak{g} . Once one takes the step to transforming automorphic forms, the freedom is much bigger. Remember that the scalar degrees of freedom reside in the coset $G(\mathbb{Z}) \backslash G/K$, and that they appear through the “physical” part P of the Maurer–Cartan form $g^{-1}dg$, $P = \Pi_{\mathfrak{g}/\mathfrak{k}}(g^{-1}dg)$. Any higher-derivative term (considering purely scalar terms) contain a number of P ’s, perhaps with covariant derivatives, contracted with something that cancels the K transformation of P in the appropriate way. Note that $\Gamma(\mu, g)$ transforms correctly, so that a K -invariant object may be formed by contracting P ’s either with each other, or with Γ ’s. Again, summation over $G(\mathbb{Z})$ is of course needed. We arrive at automorphic forms of the generic form

$$\phi_{\alpha_1 \dots \alpha_k}^{(i, w, k)}(g) = \sum_{\mu \in G(\mathbb{Z})} \Gamma_{\alpha_1} \dots \Gamma_{\alpha_k} C_i(\Gamma)^{-w} , \quad (\text{A.3})$$

where again $\Gamma_\alpha = \Gamma_\alpha(\mu, g) = [\Pi_{\mathfrak{g}/\mathfrak{k}}(g^{-1}\mu g)]_\alpha$. The automorphic form ϕ is symmetric in the α_k indices. The restricted Casimir C_i is inserted for convergence of the sum. We see that, for a given choice of C_i and w , there is one automorphic form for each irreducible K -module in the symmetric tensor product of n elements in $\mathfrak{g}/\mathfrak{k}$, generically a much larger number than the number of invariant automorphic forms.

Also here, the simpler construction for $G = SL(n)$ is obtained with an even number of $(mg)_I$ ’s (I is the fundamental index) as

$$\psi_{I_1 \dots I_{2l}}^{(w, l)}(g) = \sum_{\mathbb{Z}^n \setminus 0} (mg)_{I_1} \dots (mg)_{I_{2l}} |mg|^{-2w} . \quad (\text{A.4})$$

We would like to comment on the transformation properties of the transforming automorphic forms. As they are written (as functions of g), they are a collection of functions

on $G(\mathbb{Z}) \backslash G$, transforming under K transformations as specified by the index structure. If we, on the other hand, view g as a representative of the right coset G/K by fixing a gauge encoded in some parametrisation $g = g(\tau)$, the picture changes. The coset coordinates τ transform non-linearly under $G(\mathbb{Z})$, and a compensating gauge transformation is required to get back on the gauge hypersurface. The transformations under $G(\mathbb{Z})$ are of the form $g(\tau) \rightarrow \gamma g(\tau) k(\gamma, \tau)$. In this picture, a $G(\mathbb{Z})$ transformation of the automorphic forms induces a K -transformation with the element $k(\gamma, \tau)$ on the appropriate module given by the index structure. This can of course be mimicked without gauge fixing by replacing the element $\gamma \otimes 1 \in G(\mathbb{Z}) \times K$ by the element $\gamma \otimes k(\gamma, \tau)$, which allows us to interpret the automorphic forms as collections of functions on G/K with a specific non-linear transformation property under $G(\mathbb{Z})$.

We are sometimes interested in certain limiting values of automorphic forms. In the present paper, the terms obtained after dimensional reduction should correspond to leading terms in an asymptotic expansion at large volume of a torus. We consider the possibility of collecting the terms we obtain in sums of automorphic functions of $SL(n+1)$ after reducing on T^n . With the embedding of $GL(n)$ in $SL(n+1)$ discussed earlier in section 3, where e^ϕ is the determinant of the metric on T^n , the $SL(n+1)$ group element may (with a partial gauge choice) be parametrised as

$$G = \begin{bmatrix} e^{-\phi} & 0 \\ e^{-\phi} u & e^{\frac{\phi}{n}} \tilde{e} \end{bmatrix}, \quad (A.5)$$

where \tilde{e} is a group element of $SL(n)$ parametrising the shape of T^n . The large-volume limit is $\phi \rightarrow \infty$. The shape of T^n should be irrelevant in this limit, as long as it is non-degenerate, and we take $\tilde{e} = \mathbb{1}$. An automorphic form of the type in eq. (A.4) with $2w > 2l + 1$ (for convergence) is then dominated by terms with $m = (m_0, 0, \dots, 0)$ and has the limiting value [10]

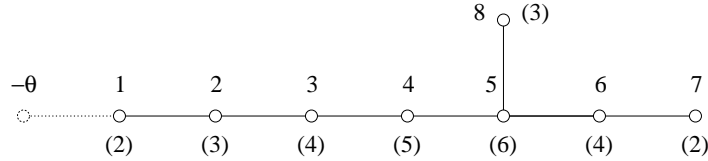
$$\psi_{I_1 \dots I_{2l}}^{(w,l)}(g) \xrightarrow{\phi \rightarrow \infty} e^{2(w-l)\phi} \zeta(2(w-l)) \delta_{I_1,0} \dots \delta_{I_{2l},0}. \quad (A.6)$$

Finally, it is interesting to count the number of possible terms one can write down in a concrete situation. Much of the present paper aims at reduction to $d = 3$ and the coset $E_{8(8)}(\mathbb{Z}) \backslash E_8 / (Spin(16)/\mathbb{Z}_2)$. An R^4 correction contains terms with up to eight P 's. Just considering these for a given w (*i.e.*, for the moment omitting the terms with derivatives of P), and assuming that we use the quadratic Casimir, the number of possible terms obtainable are labeled by irreducible $\mathfrak{so}(16)$ representations in the symmetric product of eight chiral spinors. The number of representations, *i.e.*, of automorphic forms, is 222. This can be compared to the number of scalars, 2, which is obtained directly from the E_8 Casimir operators. The corresponding number relevant for gravity, *i.e.*, the number of irreducible

$\mathfrak{so}(9)$ representations in the symmetric product of eight symmetric traceless tensors, is 609. It seems that demanding $E_{8(8)}$ invariance gives some restriction even on the possible $SL(9, \mathbb{Z})$ -invariant terms involving the gravitational scalars only, but it will take some ingenuity to extract the information. It is tempting to believe that the octic E_8 invariant [23] has some special rôle in the R^4 terms, but this remains unclear in the light of the large number of transforming automorphic functions.

APPENDIX B: REDUCTION OF THE OCTIC INVARIANT TO MATRICES

By assuming that E_8 organises the scalars after compactification to three dimensions also after the inclusion of R^4 terms we can obtain constraints related to $SL(9)$ which are more readily checked. To see this, consider the \mathfrak{e}_8 Dynkin diagram, with Coxeter labels and extended root:



The horizontal line consists of the simple roots of $\mathfrak{sl}(9)$. In the standard way of embedding $\mathfrak{sl}(n)$ roots in $(n + 1)$ -dimensional space, an element in the Cartan algebra of $\mathfrak{sl}(9)$ (and, thereby, of \mathfrak{e}_8) can be written in an orthonormal basis as $M = (m_0, m_1 - m_0, m_2 - m_1, \dots, m_7 - m_6, -m_7)$. We have $\alpha_0 = -\theta = -(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8)$. Solving for α_8 gives $\alpha_8 = \frac{1}{3}(-1, -1, -1, -1, -1, -1, 2, 2, 2)$ in the orthonormal basis.

Invariants under $\mathfrak{sl}(9)$ restricted to the CSA can be formed as $\text{tr} M^n \equiv \sum_{i=1}^9 (M_i)^n$ (*i.e.*, the vector M above is thought of as the diagonal of a matrix M). They will all be automatically invariant under the Weyl group of $\mathfrak{sl}(9)$, generated by simple reflections permuting nearby components of the 9-dimensional vectors in the orthonormal basis. The only thing one has to check for invariance under the Weyl group of \mathfrak{e}_8 is invariance under reflection in the hyperplane orthogonal to the exceptional root α_8 . As a (9×9) -matrix it is

realised as

$$w(\alpha_8) = \mathbb{1} - \alpha_8^t \alpha_8 = \frac{1}{9} \begin{bmatrix} 8 & -1 & -1 & -1 & -1 & -1 & 2 & 2 & 2 \\ -1 & 8 & -1 & -1 & -1 & -1 & 2 & 2 & 2 \\ -1 & -1 & 8 & -1 & -1 & -1 & 2 & 2 & 2 \\ -1 & -1 & -1 & 8 & -1 & -1 & 2 & 2 & 2 \\ -1 & -1 & -1 & -1 & 8 & -1 & 2 & 2 & 2 \\ -1 & -1 & -1 & -1 & -1 & 8 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 5 & -4 & -4 \\ 2 & 2 & 2 & 2 & 2 & 2 & -4 & 5 & -4 \\ 2 & 2 & 2 & 2 & 2 & 2 & -4 & -4 & 5 \end{bmatrix}, \quad (B.1)$$

and acts on M as $w(\alpha_8)M = M + \frac{1}{3}m_5(-1, -1, -1, -1, -1, -1, 2, 2, 2)$. A general Ansatz for the restriction of the octic \mathfrak{e}_8 invariant to the CSA (using $\mathfrak{sl}(9)$ “covariance”) is $S(M) = \text{tr}M^8 + a\text{tr}M^6\text{tr}M^2 + b\text{tr}M^5\text{tr}M^3 + c(\text{tr}M^4)^2 + d\text{tr}M^4(\text{tr}M^2)^2 + e(\text{tr}M^3)^2\text{tr}M^2 + f(\text{tr}M^2)^4$. The coefficient f is of course arbitrary, and will be left out. We demand that $S(w(\alpha_8)M) = S(M)$. A short Mathematica calculation then gives the values of the coefficients in the Ansatz:

$$S(M) = \text{tr}M^8 - \frac{28}{45}\text{tr}M^6\text{tr}M^2 - \frac{28}{45}\text{tr}M^5\text{tr}M^3 - \frac{7}{36}(\text{tr}M^4)^2 + \frac{7}{36}\text{tr}M^4(\text{tr}M^2)^2 + \frac{7}{27}(\text{tr}M^3)^2\text{tr}M^2. \quad (B.2)$$

This is the polynomial (in the symmetric (9×9) -matrix \mathscr{P}) we should look for in the R^4 terms if it is multiplied by a scalar automorphic form of E_8 . It has to be the same formal expression already in terms of the P of $SO(8)$.

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